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P POINTS WITH COUNTABLY MANY CONSTELLATIONS

BY

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ABSTRACT. If the continuum hypothesis (CH) is true, then for any P point ultrafilter D (on the set of natural numbers) there exist initial segments of the Rudin-Keisler ordering, restricted to (isomorphism classes of) P points which lie above D, of order type $\aleph_1$. In particular, if D is an RK-minimal ultrafilter, then we have (CH) that there exist P-points with countably many constellations.

0. Introduction. Our main result is that in the presence of the continuum hypothesis (henceforth denoted CH), there exist P point ultrafilters on ω with exactly $\aleph_0$ many constellations. Actually, we prove a somewhat stronger theorem about initial segments of the Rudin-Keisler (RK) ordering on the class of P points; in order to state this result, we begin with a few definitions. All ultrafilters here are nonprincipal ultrafilters on $\omega = \{0, 1, 2, \dots\}$. An ultrafilter D is a *P point* iff any function $f: \omega \to \omega$ is either constant or finite-to-one on a set in D. P points have been studied extensively, and we shall assume basic results about them and their RK ordering; good references are [**B1** and **Pu**]. If D is a P point, let $<_{P,D}$ denote the RK ordering on (equivalence classes of) P points which lie above D in RK. An *initial segment* of $<_{P,D}$ means a downward closed subset, and the initial segment *determined by* E is $\{F: D \leqslant F < E\}$ (we use $<$ to denote the RK ordering).

In his thesis [**Ec**], Eck showed (CH) that if D is any P point, then there exist P points E immediately above D in RK in the strong sense that any strict RK predecessor of E is a predecessor of D; we call such an E a *strong immediate successor* ($s.i.s$) of D. Iterating Eck's theorem ω times yields the existence (CH), for any P point D, of initial segments of $<_{P,D}$ of order type ω. Our main theorem is the existence (CH) of initial segments of $<_{P,D}$ of order type $\aleph_1$; the bulk of the article is devoted to its proof.

In [**B3**], Blass proved the result just stated without the restriction to P points; that is, he showed (CH) that for any ultrafilter D, there exist initial segments of "RK above D" of order type $\aleph_1$. The proof involved reformulating the problem in model theoretic terms, and we shall take the same approach. Let **N** be the complete first order structure on ω (i.e. the language for **N** contains names for every finitary function and relation on ω). We use the term *model* to mean "nonstandard model of Th(**N**)", and we use $*f$ to indicate the interpretation of the function $f: \omega \to \omega$ in whichever model is under consideration. If D is an ultrafilter, then *D-prod* denotes

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the ultrapower of $\mathbf{N}$ by D, and if f is a function from ω to ω, then the corresponding element of the universe of D-prod (called the *germ* of f) is denoted $[f]_D$. In general, a model is isomorphic to an ultrapower iff it is finitely generated, which, due to the existence of pairing functions, is the same as saying that the model is generated by a single element in its universe.

There is an intimate relationship between the structure of an ultrapower D-prod and the RK ordering below D; roughly, finitely generated submodels of D-prod correspond to RK predecessors of D: the submodel generated by $[f]_D$ corresponds to the RK predecessor $f(D)$ of D (the submodel is isomorphic to $f(D)$-prod by the map f^*: $f(D)$-prod $\to D$-prod defined by $f^*([g]_{f(D)}) = [g \circ f]_D$). The details for this construction are well known and can be found in **[B2]**. An ultrafilter D is a P point iff every (nonstandard) submodel of D-prod is cofinal in D-prod, and in general, we refer to models in which all nonstandard submodels are cofinal as "single-skied" (see **[Pu]** for a discussion of skies). Then a model $\mathscr{A}$ of Th($\mathbf{N}$) which is single-skied and finitely generated is isomorphic to D-prod for some P point D. Note that if $f(E) = D$, then E is a s.i.s. of D iff the submodel $\mathscr{E}$ of D-prod generated by $[f]_D$ ($\mathscr{E} = f^{*\prime\prime}E$-prod) is *strictly maximal* in D-prod (that is, every proper submodel of D-prod is a submodel of $\mathscr{E}$). Also, if $\mathscr{D}$ is a *strictly minimal* extension of the model $\mathscr{E}$ (that is, $\mathscr{E}$ is strictly maximal in $\mathscr{D}$), then $\mathscr{D}$ must be isomorphic to an ultrapower, since any element of $\mathscr{D} - \mathscr{E}$ must generate $\mathscr{D}$. The notation $f''X$ means the image of the set X under the function f.

1. Extensions of countably generated models. The main result mentioned above will involve the construction of a sequence of P points $\{D_\alpha: \alpha < \aleph_1\}$ for any P point D, with $D_0 = D$, which form the desired initial segment of $<_{P,D}$. At successor stages, we construct a P point $D_{\alpha+1}$ which is a s.i.s. of D_α with a modified version of Eck's technique; the modifications are included to make the limit stages go through. Model theoretically, the ultrapower $D_{\alpha+1}$-prod is a strictly minimal, cofinal extension of (the embedded image of) D_α-prod. The strategy at limit stages requires a few more definitions. Suppose λ is a limit ordinal and $\{D_\alpha: \alpha < \lambda\}$ is an RK-increasing sequence of ultrafilters. Call an ultrafilter E a *strongly minimal upper bound* (*s.m.u.b.*) for the given sequence if $D_\alpha < E$ for all $\alpha < \lambda$ and any strict RK predecessor of E is a predecessor of D_α for some $\alpha < \lambda$. Our construction will insure that D_λ is a P point and a s.m.u.b. for $\{D_\alpha: \alpha < \lambda\}$, and it is easy to see then that we will obtain our desired sequence. The actual construction of D_λ for countable limit ordinals λ involves an excursion into model theory, which we now describe. Suppose we have P points D_α for $\alpha < \lambda$ satisfying the description above. Let $\alpha_1, \alpha_2, \ldots$ be a cofinal ω-sequence in λ, let $E_i = D_{\alpha_i}$ and let p_i: $\omega \to \omega$ such that $p_i(E_{i+1}) = E_i$. Then p_i^* embeds E_i-prod into E_{i+1}-prod, and so we can form the direct limit $\mathscr{A}$ of the system $\{\langle E_i\text{-prod}, p_i^*\rangle: i = 1, 2, \ldots\}$; let $\mathscr{E}_i$ be the canonical image of E_i-prod in $\mathscr{A}$, so that $\mathscr{A}$ is the union of the $\mathscr{E}_i$. Now $\mathscr{A}$ is a model of Th($\mathbf{N}$) and $\mathscr{A}$ is single-skied since each of the E_i are P points and hence the models $\mathscr{E}_i$ are mutually cofinal. Note that $\mathscr{A}$ is not finitely generated and hence not isomorphic to an ultrapower.

Suppose that $\mathscr{A}$ admits a strictly minimal, cofinal extension $\mathscr{B}$. Then $\mathscr{B}$ must be (isomorphic to) an ultrapower, and $\mathscr{B}$ must be single-skied since any proper submodel of $\mathscr{B}$ is a submodel of, and hence cofinal in, $\mathscr{A}$ and $\mathscr{A}$ is cofinal in $\mathscr{B}$. Thus $\mathscr{B}$ is isomorphic to F-prod for some P point F, and we set $D_\lambda = F$. It is easy to check that D_λ is a s.m.u.b. for $\{D_\alpha: \alpha < \lambda\}$.

The discussion above shows that we can succeed at limit stages of our construction if we can find a strictly minimal, cofinal extension $\mathscr{B}$ of the countably generated model $\mathscr{A}$ which arises as a direct limit of previously constructed models. In **[B3]**, Blass proved a characterization of those countably generated models of Th(**N**) which admit strictly minimal extensions. We shall require a number of modifications to that theorem, and what follows, through the proof of Theorem 2, is adapted from **[B3]**.

Let $\in'$ be the binary relation on ω defined by $m \in' n$ iff 2^m occurs in the binary expansion of n, so n codes the finite set $\{m: m \in' n\}$. If $\mathscr{A}$ is a model and $a \in \mathscr{A}$, then let $a(\mathscr{A}) = \{b \in \mathscr{A}: \mathscr{A} \models (b \in' a)\}$. If $a \in \mathscr{A} \prec \mathscr{B}$, then $a(\mathscr{A}) = a(\mathscr{B}) \cap \mathscr{A}$. Assume that the set of finite sequences from ω has been coded in some standard way, and let $\langle\ ,\ ,\dots\rangle$ denote the coding function. Let Seq be the set of codes, and for each x in Seq, $\mathrm{lh}(x)$ is the length of x and $(x)_k$ is the kth component of x if $k < \mathrm{lh}(x)$ and $(x)_k = 0$ otherwise. Blass's result is

THEOREM 1 (BLASS **[B3]**) (CH). *Let $\mathscr{A}$ be a countably generated model of* Th(**N**). *$\mathscr{A}$ admits a strictly minimal extension if and only if for any sequence $\{a_i \in \mathscr{A}: i \in \omega\}$ with $a_i(\mathscr{A})$ nonempty and $a_0(\mathscr{A}) \supseteq a_1(\mathscr{A}) \supseteq a_2(\mathscr{A}) \supseteq \cdots$, either*

(i) $\bigcap_{i \in \omega} a_i(\mathscr{A}) \neq N\varnothing$, *or*

(ii) *for any b in $\mathscr{A}$, there is a c in $a_0(\mathscr{A})$ and an $f: \omega \to \omega$ such that $^*f(c) = b$.*

REMARKS. The "only if" direction does not use CH. If $\mathscr{A}$ is finitely generated, and hence isomorphic to an ultrapower, then (i) always holds since ultrapowers are $\aleph_1$-saturated **[CK**, p. 305], and so (CH) ultrapowers always admit strictly minimal extensions.

Our first modification takes care of insuring that the new model is a cofinal extension.

THEOREM 2 (CH). *Let $\mathscr{A}$ be a single-skied, countably generated model of* Th(**N**), *and suppose $\mathscr{A}$ satisfies the conditions of Theorem* 1. *Then $\mathscr{A}$ admits a single-skied, strictly minimal extension.*

PROOF. Most of this proof is identical to the proof of Theorem 1 given in **[B3]**. First, if $\mathscr{A}$ is finitely generated, and hence isomorphic to a P point ultrapower, then this theorem is simply Eck's result that any P point has strong immediate successors which are P points. Assume therefore that $\mathscr{A}$ is not finitely generated, and let $\{a_n: n = 1, 2, \dots\}$ be a set which generates $\mathscr{A}$; without loss of generality, the sequence $\langle a_n: n = 1, 2, \dots\}\rangle$ is not redundant, that is, a_{n+1} is not in the submodel of $\mathscr{A}$ generated by $^*\langle a_1, a_2, \dots, a_n\rangle$, and let $g_n = {}^*\langle a_1, \dots, a_n\rangle$. Let TR_n: Seq $\to$ Seq be the map which truncates sequences by removing all but the first n components (and leaves shorter sequences fixed). Then $^*\mathrm{TR}_n(g_m) = g_n$ for all $m > n$; note also that

g_m is not in the submodel of $\mathscr{A}$ generated by g_n if $n < m$ (by the nonredundancy of the a_i's). Let $\mathscr{G}_n$ be the submodel generated by g_n.

Let G_n be the type of g_n in $\mathscr{A}$, that is $G_n = \{X \subseteq \text{Seq}: \mathscr{A} \models {}^*X(g_n)\}$. Then G_n is an ultrafilter on Seq, in fact a P point (since $\mathscr{A}$, and hence each of its submodels, is single-skied), and $\text{TR}_n(G_m) = G_n$ for $m > n$. G_n concentrates on sequences of length n, that is $\{x \subseteq \text{Seq}: \text{lh}(x) = n\} \in G_n$, and the nonredundancy implies that TR_n is not one-to-one on any set in G_m for $m > n$.

To obtain a strictly minimal, single-skied extension of $\mathscr{A}$, it suffices to construct an ultrafilter E on Seq such that

(1) for all $n \geqslant 1$, $\text{TR}_n(E) = G_n$,

(2) for all f: Seq $\to \omega$, there is a set A in E such that either f is one-to-one on A or, for some n, f is TR_n-fiberwise constant on A (that is, f is constant on sets of the form $A \cap \text{TR}_n^{-1}\{i\}$), and

(3) for some set B in E, TR_1 is finite-to-one on B.

Given such an ultrafilter E, we can embed $\mathscr{A}$ into E-prod by mapping g_n to $[\text{TR}_n]_E$ (by (1)), and for simplicity we identify A with its embedded image in E-prod. By (2), every element of E-prod either generates E-prod or is in the submodel $\mathscr{G}_n$ for some n, and so E-prod is a strictly minimal extension of $\mathscr{A}$. By (3), the submodel $\mathscr{G}_1$, and hence $\mathscr{A}$, is cofinal in E-prod. It follows that E-prod is single-skied since any proper submodel of E-prod is a submodel of (and hence cofinal in) $\mathscr{A}$, and $\mathscr{A}$ is cofinal in E-prod.

The existence proof for E is a typical sort of inductive construction for ultrafilters on ω. Call a subset L of Seq *large* if $\text{TR}_n''L \in G_n$ for all n; otherwise L is *small*. Any ultrafilter consisting entirely of large sets satisfies (1). Thus it suffices to construct a filter F consisting of large sets, such that F contains a set B satisfying (3) and for each f: Seq $\to \omega$, F contains a set A satisfying (2). Then let E be any ultrafilter extending F and containing the complements of all small sets. To construct F, first order ${}^{\text{Seq}}\omega$ in an $\aleph_1$-sequence (by CH) and then inductively define large sets L_α for $\alpha < \aleph_1$ such that L_0 satisfies (3), $L_{\alpha+1}$ works as A in (2) for the αth function f and $L_\alpha - L_\beta$ is small for $\alpha \geqslant \beta$. It is easy to check that the finite union of small sets is small, and it follows that $\{L_\alpha: \alpha < \aleph_1\}$ generates a filter; let F be this filter.

To construct L_0, first find a set B_n in G_n for $n \geqslant 1$ such that B_n consists of sequences of length (exactly) n, TR_1 is finite-to-one on B_n and $\text{TR}_1''B_n \subseteq \{n, n+1, n+2, \ldots\}$. Such B_n exist since the G_n's are P points (and G_1 is nonprincipal). Set $L_0 = \bigcup_{n \geqslant 1} B_n$, and then $\text{TR}_n''L_0$ includes B_n for all n, so L_0 is large. For any $k \geqslant 1$, $\text{TR}_1^{-1}\langle k \rangle$ is the union of k finite sets, so TR_1 is finite-to-one on L_0.

The construction of L_λ for limit λ uses only that finite unions of small sets are small. The successor stages use the hypotheses of the theorem (that is, the conditions on $\mathscr{A}$ given in Theorem 1) to construct a large subset A of L which satisfies (2) for any large set L and any f: Seq $\to \omega$. The details for both the limit and successor cases can be found in [**B3**, pp. 154–155].

By Theorem 2, we will succeed at limit stages in the construction of our sequence $\{D_\alpha: \alpha < \aleph_1\}$ if the associated (countably generated) model satisfies the conditions of Theorem 1. The model $\mathscr{A}$ arising in our construction has a special structure in that those submodels of $\mathscr{A}$ which include (the embedded copy of) D_0-prod are linearly (in fact, well) ordered by inclusion. This makes it somewhat easier to satisfy the conditions of Theorem 1, as the next two lemmas show.

LEMMA 3 (CH). *Suppose* $\mathscr{E}_1 \precneqq \mathscr{E}_2 \precneqq \mathscr{E}_3 \precneqq \cdots$ *is an ascending chain of countably generated models of* Th(**N**) *such that*

(a) *each* $\mathscr{E}_n$ *admits a strictly minimal extension*,

(b) *for any* $b \in (\mathscr{E}_{n+1} - \mathscr{E}_n)$, *the submodel generated by* b *includes* $\mathscr{E}_n$, *and*

(c) $\mathscr{A} = \bigcup_{n \geqslant 1} \mathscr{E}_n$ *does not admit a strictly minimal extension.*

Then there is an element $a \in \mathscr{A}$ *and some* $J \in \omega$ *such that* $a(\mathscr{A}) \subseteq \mathscr{E}_J$ *but* $a \notin \mathscr{E}_J$.

PROOF. First note that (b) says that for any b in $\mathscr{E}_{n+1} - \mathscr{E}_n$ and any c in $\mathscr{E}_n$, there is an $f\colon \omega \to \omega$ with ${}^*f(b) = c$. By Theorem 1, there is a sequence $\{a_i\}$ such that $a_i(\mathscr{A}) \supseteq a_{i+1}(\mathscr{A})$, $a_i(\mathscr{A}) \neq \varnothing$, $\bigcap_{i \geqslant 0} a_i(\mathscr{A}) = \varnothing$ and $a_0(\mathscr{A})$ does not generate $\mathscr{A}$ by standard unary functions. Then, for some J, $a_0(\mathscr{A}) \subseteq \mathscr{E}_J$, since otherwise, for arbitrarily large k, $a_0(\mathscr{A}) \cap (\mathscr{E}_{k+1} - \mathscr{E}_k)$ is nonempty, and then it follows from (b) that every element of $\mathscr{A}$ is obtainable from an element of $a_0(\mathscr{A})$ by a standard unary function, thus contradicting the choice of $\{a_i\}$. We have then that, for all i, $a_i(\mathscr{A}) \subseteq \mathscr{E}_J$, and if a_i is an element of $\mathscr{E}_J$ then $a_i(\mathscr{E}_J) = a_i(\mathscr{A})$.

We now claim that for some i, a_i is not an element of $\mathscr{E}_J$ (and then the proof is complete by setting $a = a_i$). Suppose not; then for all i, a_i is in $\mathscr{E}_J$ and so a_i is also in $\mathscr{E}_{J+1}$. Since $\mathscr{E}_{J+1}$ admits a strictly minimal extension, and since $a_i(\mathscr{E}_{J+1}) = a_i(\mathscr{A}) \supseteq a_{i+1}(\mathscr{A}) = a_{i+1}(\mathscr{E}_{J+1})$, we have by Theorem 1 that either $a_0(\mathscr{E}_{J+1})$ generates $\mathscr{E}_{J+1}$ or $\bigcap_{i \geqslant 0} a_i(\mathscr{E}_{J+1}) = \varnothing$. The latter conclusion is impossible by the choice of $\{a_i\}$ and the fact that $a_i(\mathscr{A}) \subseteq \mathscr{E}_{J+1}$, and the former conclusion says that a subset of $\mathscr{E}_J$ generates $\mathscr{E}_{J+1}$, which is impossible since $\mathscr{E}_J$ is a proper submodel of $\mathscr{E}_{J+1}$.

Let $\mathscr{M}$ be a finitely generated model. We say that $\mathscr{M}$ is *element generated* iff for every generator a of $\mathscr{M}$, there is a generator b of $\mathscr{M}$ with $b \in a(\mathscr{M})$.

LEMMA 4 (CH). *Suppose that* $\{\mathscr{E}_i\colon i = 1, 2, \ldots\}$ *form an ascending chain of countably generated models*, $\mathscr{A} = \bigcup_{i \geqslant 1} \mathscr{E}_i$, *and that all the hypotheses of Lemma* 3 *are satisfied. Then there is a finitely generated submodel* $\mathscr{M}$ *of* $\mathscr{A}$ *which includes* $\mathscr{E}_1$ *and is not element generated.*

PROOF. Let a and $\mathscr{E}_J$ satisfy the conclusion of Lemma 3, and let $\mathscr{M}$ be the submodel of $\mathscr{A}$ generated by a. Then $\mathscr{E}_1 \subseteq \mathscr{E}_J \subsetneqq \mathscr{M}$ (the second inclusion follows from hypothesis (b) of Lemma 3), and $\mathscr{M}$ is not element generated since $a(\mathscr{M}) \subseteq \mathscr{E}_J$ and so $a(\mathscr{M})$ does not contain a generator of $\mathscr{M}$.

2. Simple combinatorics. Let $\langle A_1, A_2, \ldots, A_n \rangle$ be a sequence of nonempty sets. A *complete set of distinct representatives* (*CDR*) is the image of a one-to-one choice function on the set $\{A_1, \ldots, A_n\}$, that is, a sequence $\langle a_1, \ldots, a_n \rangle$ with $a_i \in A_i$ and $a_i \neq a_j$ if $i \neq j$. We will need the following simple combinatorial lemma in order to construct ultrapowers which are element generated.

LEMMA 5. *Let* $\langle A_0, \ldots, A_{n-1} \rangle$ *be a sequence of distinct, nonempty sets. Then there is a subsequence of length at least* $\log_2(n)$ *which admits a CDR.*

PROOF. The lemma is obvious by inspection for $n < 5$, so assume $n \geqslant 5$. Let $K = |\bigcup_{0 \leqslant i < n} A_i|$. We prove the lemma by induction on K. Since $n > 4$, we have $K > 1$, so assume the lemma for smaller K, and fix an arbitrary $x \in \bigcup A_i$. Reorder the A_i so that $x \in A_i$ for $i < p$ and $x \notin A_i$ for $i \geqslant p$, where p is the number of sets among the A_i which contain x. Then $0 < p \leqslant n$.

Case 1: $0 < p < (n/2)$. In this case, $\langle A_p, A_{p+1}, \ldots, A_{n-1} \rangle$ satisfies the induction hypothesis, so there is a subsequence $\langle B_0, \ldots, B_{m-1} \rangle$ of $\langle A_p, \ldots, A_{n-1} \rangle$ which admits a CDR $\langle b_0, \ldots, b_{m-1} \rangle$, and $m \geqslant \log_2(n-p) \geqslant \log_2(n/2) = \log_2(n) - 1$. Then $\langle x, b_0, \ldots, b_{m-1} \rangle$ is a CDR for $\langle A_0, B_0, \ldots, B_{m-1} \rangle$, which has length $m + 1 \geqslant \log_2(n)$.

Case 2: $(n/2) \leqslant p \leqslant n$. For $i < p$, let $B_i = (A_i - \{x\})$. By the induction hypothesis, there is a CDR $\langle d_0, \ldots, d_{m-1} \rangle$ for some subsequence of the B_i's with $m \geqslant \log_2(p) \geqslant \log_2(n) - 1$; without loss of generality, assume the A_i were ordered so that the subsequence of the B_i's admitting this CDR is $\langle B_0, \ldots, B_{m-1} \rangle$. If $m \geqslant \log_2(n)$, we are done; otherwise, since $n > 4$ we have $p \geqslant n/2 \geqslant \log_2(n) > m$. Thus $\langle d_0, \ldots, d_{m-1}, x \rangle$ is a CDR for $\langle A_0, \ldots, A_{m-1}, A_{p-1} \rangle$.

3. The successor case. The point of the theorem in this section is to insure the existence of a P point $D_{\alpha+1}$ which is a s.i.s. of D_α and such that $D_{\alpha+1}$-prod is element generated. If E is an ultrafilter, then the generators of E-prod are the germs $[f]_E$ of one-to-one (mod E) functions f, and so it follows that E-prod is element generated iff for any such one-to-one germ $a = [f]_E$, there is a one-to-one (mod E) function g with $[g]_E \in a(E\text{-prod})$, which means there is a set A in E such that for all x in A, $g(x) \in' f(x)$.

The construction of $D_{\alpha+1}$ will be done within the framework of Theorem 2.2 of [**Ro**]. For the convenience of the reader we state this theorem below after supplying the requisite definitions. If X is a set and p is a function which is finite-to-one on X, then the *cardinality function* of X with respect to p, denoted $C_{X,p}$, is defined by $C_{X,p}(n) = |X \cap p^{-1}\{n\}|$. We will omit reference to p when there is no ambiguity. A (*Dedekind*) *cut* in an ultrapower D-prod is a partition of D-prod into convex sets S and L such that every element of S precedes every element of L. A cut is *fair* if S and L are nonempty and L has no countable coinitial subset. If E is a P point and p: $\omega \to \omega$ with $p(E) = D$, then the cut in D-prod *associated to* p and E is defined by putting into L those D-germs $[C_{X,p}]_D$ for $X \in E$ (and all larger D-germs), and setting $S = D\text{-prod} - L$ (see [**B2**] for a thorough discussion of Dedekind cuts in ultrapowers). Finally, a *condition* on X is simply a statement about X.

THEOREM 6 [**Ro**] (CH). *Let D be a P point, $\langle S, L \rangle$ a fair cut in D-prod such that S is closed under addition in D-prod, p the first projection from ω^2 to ω and $\{C_i \colon i \in I\}$ a set of at most $2^{\aleph_0}$ conditions on subsets of ω^2. Call a set $X \subseteq \omega^2$ large if it contains a subset Y on which p is finite-to-one and $[C_Y]_D$ is in L, and suppose that for any large X and condition C_i, there is a large subset Y of X which satisfies C_i.*

Then there exist $2^{\aleph_1}$ many (pairwise nonisomorphic) P points E on ω^2 with $p(E) = D$ and associated cut $\langle S, L \rangle$, such that for all i in I, E contains a set satisfying condition C_i.

PROOF. This is a special case of Theorem 2.2 in **[Ro]** obtained by taking the fiber measure of that theorem to be the cardinality function (see also **[B2]**).

THEOREM 7 (CH). *Let D be a P point, p the first projection from ω^2 to ω and $[h]_D$ any element of D-prod. There exist $2^{\aleph_1}$ P points E with $p(E) = D$ such that*

(1) E is a s.i.s. of D,

(2) E-prod is element generated, and

(3) $[h]_D$ is in the small part S of the cut in D-prod associated to p and E.

PROOF. Begin by defining functions $s_j\colon \omega \to \omega$ as follows;

$$s_0(0) = h(0),$$
$$s_{i+1}(k) = 2^{s_i(k)},$$
$$s_0(k+1) = \text{maximum of } \left\{ h(k+1),\ \sum_{j<k} s_j(j) \right\}.$$

Define a cut $\langle S, L\rangle$ in D-prod by $a \in S$ iff $a \leqslant [s_j]_D$ for some j. Then $[h]_D \in S$ and $\langle S, L\rangle$ is a fair cut since the existence of a countable cofinal set in S implies that there is no countable coinitial set in L (by the $\aleph_1$-saturation of D-prod). It is easy to check that S is closed under addition, multiplication and exponentiation. For each $f\colon \omega^2 \to \omega$, let C_f be the following condition on Y: (f is p-fiberwise constant on Y) or (f is one-to-one on Y and there is a p-fiberwise one-to-one function g on Y such that for all $y \in Y$, $g(y) \in' f(y)$).

If we show that, for any f, any large set X includes a large subset Y satisfying C_f, then by Theorem 6 we will have shown the existence of $2^{\aleph_1}$ many P points E with $p(E) = D$, associated cut $\langle S, L\rangle$, such that E contains sets satisfying C_f for all f. Thus every f is either p-fiberwise constant or (globally) one-to-one on a set in E; so every $[f]_E$ in E-prod either is in the embedded image (by p^*) of D-prod or is a generator of E-prod. It follows that E is a s.i.s. of D. To see that E-prod is element generated, let a be a generator of E-prod; so $a = [f]_E$ for some function f which is one-to-one on a set in E. Let $Y \in E$ satisfy C_f; then f cannot be p-fiberwise constant on Y (since $[f]_E$ is a generator of E-prod), so Y satisfies the second part of C_f. Then there is a p-fiberwise one-to-one function g on X such that for all $y \in Y$, $g(y) \in' f(y)$, and so $[g]_E$ is in $a(E$-prod). Now $[g]_E$ is not in $p^{*\prime\prime}D$-prod since if it were, then g would be p-fiberwise constant as well as p-fiberwise one-to-one on a set in E, which would imply that p is one-to-one (mod E), contradicting the fairness of the associated cut. By the strict maximality of $p^{*\prime\prime}$-prod, it follows then that $[g]_E$ is a generator of E-prod. Thus E-prod is element generated.

It remains to show that we can find a large subset satisfying any given condition C_f for any large X. We can assume (by cutting down X if necessary) that p is finite-to-one on X. For each nonempty fiber $X_n = (X \cap p^{-1}\{n\})$, we can find a set $Z_n \subseteq X_n$ such that f is constant on Z_n or f is one-to-one on Z_n and $|Z_n|^2 \geqslant |X_n|$. Partition the fibers into sets W_c and W_1, where W_c consists of those n such that f is constant on Z_n and W_1 consists of those n such that f is one-to-one on Z_n. One of these sets is in D (else X was not large). If W_c is in D, then let $Y = \bigcup_{n \in W_c} Z_n$ and clearly f is p-fiberwise constant on Y; for all n in W_c, $(C_Y(n))^2 \geqslant C_X(n)$, and so $[C_Y]_D$ is in L since $[C_X]_D$ is in L and S is closed under multiplication.

If, on the other hand, W_1 is in D, then we must cut down $Z = \bigcup_{n \in W_1} Z_n$ to a large Y on which f is one-to-one and for which we can find the desired function g. First define sets B_i in D ($i = 1, 2, 3, \ldots$) by

$$B_i = \{m \in p''Z\colon |Z_m| \geqslant s_{i+1}(m)\} - \{0, 1, \ldots, m\}.$$

Then B_i is in D since Z is large (by the argument given above for Y) and hence, for all j, $C_Z(m)$ cannot be less than $s_j(m)$ for D-many m. Note also that $B_{i+1} \subseteq B_i$, and we may assume that $p''Z \subseteq B_0$ (throw away some fibers of Z if necessary). Define $t(m)$ = maximum i such that $m \in B_i$ for m in B_0. Then, for all $m \in B_0$, $t(m) < m$, $m \in B_{t(m)}$, and $|Z_m| \geqslant s_{t(m)+1}(m)$. Let M be the minimal element of B_0, and let Q_M be any subset of Z_M with $|Q_M| = s_{t(M)}(M)$. Note that f is one-to-one on Q_M.

Let $m \in B_0$, let $H_m = \{j \in B_0\colon j < m\}$, and suppose, as an induction hypothesis, that for all j in H_m, we have defined $Q_j \subseteq Z_j$ such that $|Q_j| = s_{t(j)}(j)$ and f is one-to-one on $(\bigcup_{j \in H_m} Q_j)$. Let $R_m = \{x \in Z_m\colon f(x) = f(y)$ for some y in $(\bigcup_{j \in H_m} Q_j)\}$. Since f is one-to-one on Z_m, we have

$$|R_m| \leqslant \sum_{j \in H_m} |Q_j| = \sum_{j = H_m} s_{t(j)}(j) \leqslant \sum_{j < m} s_j(j) \leqslant s_0(m),$$

and thus

$$|Z_m - R_m| \geqslant s_{t(m)+1}(m) - s_0(m) = 2^{s_{t(m)}(m)} - s_0(m) \geqslant s_{t(m)}(m).$$

Let Q_m be any subset of $Z_m - R_m$ of size $s_{t(m)}(m)$; this completes the induction construction of the Q_m for m in B_0, for it is clear by the definition of R_m that f is one-to-one on $(\bigcup_{j \in H_m} Q_j) \cup Q_m$. Let $Q = \bigcup_{j \in B_0} Q_j$; then f is one-to-one on Q and for any k in ω, we have that for all m in B_k, $|Q_m| = s_{t(m)}(m) \geqslant s_k(m)$, and so $[C_Q]_D \geqslant [s_k]_D$. Thus Q is large. The argument just given to make f one-to-one on a large set is due to Eck **[Ec]**.

If f assumes the value 0 on Q, then remove that point from Q. Temporarily fix $m \in B_0 = p''Q$, and write Q_m $(= p^{-1}\{m\} \cap Q)$ as $\{a_1, a_2, \ldots, a_n\}$. Let $A_i = \{t\colon t \in{}' f(a_i)\}$; since f is one-to-one on Q, $\{A_i\colon 1 \leqslant i \leqslant n\}$ is a collection of distinct nonempty sets, and so by Lemma 5, there is a subsequence $\langle B_1, \ldots, B_J\rangle$ of $\langle A_1, \ldots, A_n\rangle$ which admits a CDR $\langle u_1, \ldots, u_J\rangle$ with $J \geqslant \log_2(n)$. Reorder the $\{a_i\}$ so that $\langle u_1, \ldots, u_J\rangle$ is a CDR for $\langle A_1, \ldots, A_J\rangle$, and let $Y_m = \{a_1, \ldots, a_J\}$. Define g on Y_m by $g(a_i) = u_i$, so g is one-to-one on Y_m.

Let $Y = \bigcup_{m \in B_0} Y_m$; clearly Y satisfies condition C_f and so we only need show that Y is large. For all m in B_0, $|Y_m| \geqslant \log_2(|Q_m|)$, whereupon the largeness of Y follows from the largeness of Q and the closure of S under exponentiation.

4. The limit case. In order to apply Lemma 4 at limit stage λ in the construction of our sequence of P points, we need to know that (beyond some point in the sequence) each finitely generated submodel of the associated countably generated model $\mathscr{A}$ is element generated. The previous theorem insures this for finitely generated models arising at successor stages in the construction; the following theorem gives us this result for those arising at limit stages previous to λ.

THEOREM 8 (CH). *Let* $\{E_i: i = 1, 2, \ldots\}$ *be an RK-increasing sequence of P points with* $p_i(E_{i+1}) = E_i$, *and let* $\mathscr{A}$ *be the direct limit of* $\{\langle E_i\text{-prod}, p_i^*\rangle\}$. *For* $i \geqslant 2$, *let* $q_i = p_1 \circ p_2 \circ \cdots \circ p_{i-1}$ *and let* $\langle S^i, L^i\rangle$ *be the cut in* E_1*-prod associated to* q_i *and* E_i. *Assume that*:

(a) $\mathscr{A}$ *admits a strictly minimal extension*,

(b) $S^i \subsetneq S^{i+1}$ *for all* $i \geqslant 2$, *and*

(c) E_i*-prod is element generated for all* $i \geqslant 1$.

Then $\mathscr{A}$ *admits a single-skied, element generated, strictly minimal extension.*

PROOF. The proof is a modification of the proof of Theorem 2. Let $\mathscr{E}_i$ be the canonical image of E_i-prod in $\mathscr{A}$, and let a_i be a generator of $\mathscr{E}_i$ in $\mathscr{A}$. Let g_i and G_i be as in the proof of Theorem 2 (i.e. $g_i = {}^*\langle a_1, \ldots, a_i\rangle$ and the ultrafilter G_i is the type of g_i in $\mathscr{A}$). Since the ultrafilters E_i form an increasing RK-sequence, it follows that g_i generates $\mathscr{E}_i$ and hence $G_i = E_i$; thus G_i-prod is element generated and if $\langle S_i, L_i\rangle$ is the cut in G_1-prod associated to TR_1 and G_i, then $S_i \subsetneq S_{i+1}$. Since $\mathscr{A}$ admits a strictly minimal extension, the hypotheses of Theorem 1 are satisfied.

For each f: Seq $\to \omega$, let C_f be the following condition on a subset X of Seq: (f is TR_n-fiberwise constant on X for some n) or (f is one-to-one on X and there is a TR_1-fiberwise one-to-one function g on X such that $g(x) \in {}'f(x)$ for all x in X).

As in Theorem 2, we will construct a filter F on Seq consisting of large (in the sense of Theorem 2) sets such that F contains a set on which TR_1 is finite-to-one as well as sets satisfying C_f for each f; let E be an ultrafilter including F and the complements of all large sets. Let $\mathscr{B} = E$-prod, and then $\mathscr{B}$ is a single-skied, strictly minimal extension of (the embedded image of) $\mathscr{A}$; in addition, $\mathscr{B}$ is element generated, since the function g of condition C_f is TR_1 (and hence TR_n for all n) fiberwise one-to-one (mod E), and thus is not TR_n-fiberwise constant (mod E) for any n, insuring (by strict minimality) that $[g]_E$ is a generator of E-prod.

List the conditions in an $\aleph_1$-sequence. The proof proceeds exactly as for Theorem 2 (the definition of L_0 and the construction of L_λ for limit λ are identical), except that we must satisfy the stronger condition of the present theorem at successor stages. So assume that L is large and let C_f be the αth condition. Since the hypotheses of Theorem 1 are satisfied by $\mathscr{A}$, we can find a large $A \subseteq L_\alpha$ such that f is one-to-one on A, or, for some n, f is TR_n-fiberwise constant on A. If the latter, then let $L_{\alpha+1} = A$. Assume therefore that f is one-to-one on A; we must cut down A further so that we can define the desired function g.

For $n \geqslant 1$, let $R_n = \mathrm{TR}_n{}''A \cap \{x \in \text{Seq}: \mathrm{lh}(x) = n\}$. Since A is large, R_n is in G_n. Let $B_1 = R_1$, and for each x in B_1, let $s(x, 1) \in A$ such that $\mathrm{TR}_1(s(x, n)) = x$. Assume we have defined B_j for $j < K$ such that $B_j \subseteq R_j$, $B_j \in G_j$, with $\mathrm{TR}_j{}''B_{j+1} \subseteq B_j$, and we have defined a function s such that $s(x, j) \in A$ and $\mathrm{TR}_j(s(x, j)) = x$ for each x in B_j. Let

$$B_K = \left(R_K \cap \mathrm{TR}_{K-1}^{-1}(B_{K-1})\right) - W,$$

where $W = \{z \in R_K: z = \mathrm{TR}_K(s(x, K-1))$ for some x in $B_{K-1}\}$. It is easy to see that TR_{K-1} is one-to-one on W (two different z's which map to x under TR_{K-1} can-not both be truncations of $s(x, K-1)$), and so W is not in the ultrafilter G_K (since TR_{K-1} maps G_K to G_{K-1} and is not an isomorphism). It follows that B_K is in G_K, since both R_K and $\mathrm{TR}_{K-1}^{-1}(B_{K-1})$ are in G_K. For each x in B_K, let $s(x, K)$ be an element of A such that $\mathrm{TR}_K(s(x, K)) = x$. This completes the inductive construction of the B_K.

Let $P = \{s(x, n): n \geqslant 1$ and $x \in B_n\}$. Then $P \subseteq A$, P is large, and by the construction we have that for all s in P, there is a unique n and a unique x in B_n with $s = s(x, n)$. Define functions f_n on B_n by $f_n(x) = f(s(x, n))$. Then f_n is one-to-one on B_n since f is one-to-one on A. Since G_n-prod is element generated, we can find, for each n, a set Z_n in G_n, $Z_n \subseteq B_n$, and a one-to-one function g_n on Z_n such that for all x in Z_n, $g_n(x) \in' f_n(x)$.

Define $g: P \to \omega$ by $g(s(x, n)) = g_n(x)$. Then g is well defined, and for all s in P, $g(x) \in' f(x)$. We need only cut down P to a large $L_{\alpha+1}$ on which g is TR_1-fiberwise one-to-one.

Since $S_i \subsetneq S_{i+1}$, we can find functions h_i: $\mathrm{Seq}_1 \to \omega$ such that $[h_i] \in L_i - L_{i+1}$ (Seq_1 is the set of codes for sequences of length 1). Let $P_1 = Z_1$, and assume that we have defined sets $P_i \subseteq Z_i$ for $1 \leqslant i < K$ such that

(1) $P_i \in G_i$, and $\mathrm{TR}_{i-1}(P_i) \subseteq P_{i-1}$,

(2) for all $i \geqslant 2$, $|\mathrm{TR}_1^{-1}\langle p\rangle \cap P_i| \leqslant h_i(p)$ for all p in P_1, and

(3) $(\forall\langle p\rangle \in P_1)(\forall i, j < K)(\forall y, z \in (\mathrm{TR}_1^{-1}\langle p\rangle \cap (\bigcup_{n<K} P_n)))$ $(g_i(y) = g_j(z)$ iff $i = j$ and $y = z)$.

To define P_K, first let $V = \{z \in Z_K \cap \mathrm{TR}_{K-1}^{-1}(P_{K-1})$: $(\exists j < K)(\exists y \in P_j)(g_K(z) = g_j(y)$ and $\mathrm{TR}_1(y) = \mathrm{TR}_1(z))\}$.

If $p \in P_1$, then

$$\left|\mathrm{TR}_1^{-1}\langle p\rangle \cap V\right| \leqslant 1 + \sum_{j<K} \left|P_j \cap \mathrm{TR}_1^{-1}\langle p\rangle\right| \leqslant 1 + \sum_{j<K} h_j(\langle p\rangle)$$

(the first inequality follows since g_K is one-to-one on Z_K; the second follows from (2)). Thus $[C_V]_G \leqslant 1 + \Sigma_{j<K}[h_j]_G$, and since S_K is closed under addition (Theorem 2 of **[B2]**), it follows that $[C_V]_G \in S_K$ (as $[h_j]_G \in S_K$ for all $j < K$). Thus V is not in G_K. Since $[h_K]_G \in L_K$, we can find W in G_K such that for all $\langle p\rangle$ in P_1, $C_W(\langle p\rangle) \leqslant h_K(\langle p\rangle)$. Let

$$P_K = \left(Z_K \cap W \cap \mathrm{TR}_{K-1}^{-1}(P_{K-1})\right) - V.$$

Then P_K obviously satisfies (1) and (2), and it is easy to check, from the definition of V, that P_K satisfies (3).

We have inductively defined $\{P_j: j = 1, 2, 3, \dots\}$, and we set $L = s''(\bigcup(P_n \times \{n\}))$. Then $\mathrm{TR}_n{}''L$ includes P_n, and P_n is in G_n, so L is large. If s and t are in L with $g(s) = g(t)$ and $\mathrm{TR}_1(s) = \mathrm{TR}_1(t)$, then there are unique x, y, m, n with $x \in P_m$, $y \in P_n$, $s = s(x, m)$ and $t = s(y, n)$ (by the construction of the set P). By the definition of g, $g_m(x) = g_n(y)$, and by (3) above, $m = n$ and $x = y$, and so $s = t$. Thus g is TR_1-fiberwise one-to-one on L, and the proof is complete by setting $L_{\alpha+1} = L$.

5. The finale. We now combine the previous results to produce our desired sequence of P points.

THEOREM 9 (CH). *Let D be a P point. There exist initial segments of $\leqslant_{P,D}$ of order type $\aleph_1$.*

PROOF. We construct a sequence of P points $\{D_\alpha\colon \alpha < \aleph_1\}$ with $D_0 = D$ such that:

(1) $D_{\alpha+1}$ is a strong immediate successor of D_α,

(2) for limit $\lambda < \aleph_1$, D_λ is a strongly minimal upper bound for $\{D_\alpha\colon \alpha < \lambda\}$,

(3) D_α-prod is element generated for $\alpha \geqslant 1$, and

(4) if q_α is the map from D_α to D_1, and $\langle S_\alpha, L_\alpha\rangle$ is the cut in D_1-prod associated to q_α and D_α, then $S_\beta \subsetneq S_\alpha$ whenever $\beta < \alpha$.

By (1) and (2), such a sequence forms an initial segment of $<_{P,D}$.

If $\alpha = 1$, let D_1 be a s.i.s. of D such that D_1-prod is element generated; we can find such a D_1 by Theorem 7.

If $\alpha = \beta + 1$, fix $g\colon \omega \to \omega$ such that $[g]_{D_1}$ is in the large part L_β of the cut in D_1-prod associated to q_β and D_β. Use Theorem 7 to find an element generated P point D with $p(D_\alpha) = D_\beta$ such that D_α is a s.i.s. of D_β and such that $q_\beta^*([g]_D)$ $(= [g \circ q_\beta]_D)$ is in the small part S' of the cut in D_β-prod associated to p and D_α (q_β is the map from D_β to D_1). If A is any set in D_α, then we have that for all n in some set B in D_β, $g(q_\beta(n)) \leqslant |A \cap p^{-1}\{n\}|$. It is easy then to check that for all z in $q_\beta''B$, $|A \cap q_\alpha^{-1}\{z\}| \geqslant g(z)$ ($q_\alpha = q_\beta \circ p$ is the map from D_α to D_1); since $q_\beta''B$ is in D_1, we have that $[C_A]_{D_1} > [g]_{D_1}$, and so $[g]_{D_1}$ is in the small part S_α of the cut in D_1-prod associated to q_α and D_α. Thus $[g]_{D_1}$ witnesses that D_α satisfies (4).

If λ is a countable limit ordinal, let $1 = \alpha_1, \alpha_2, \ldots$ be an increasing ω-sequence cofinal in λ, and let $E_i = D_{\alpha_i}$. Let p_i be the map from E_{i+1} to E_i, let $\mathscr{A}$ be the direct limit of $\{\langle E_i\text{-prod}, p_i^*\rangle\}$, and let $\mathscr{E}_i$ be the canonical image of E_i-prod in $\mathscr{A}$. We claim first that $\mathscr{A}$ admits a strictly minimal extension.

To prove this claim, first note that each $\mathscr{E}_i$ admits a strictly minimal extension (since each E_i is one of the previously constructed D_α's) and that any element of $\mathscr{E}_{i+1} - \mathscr{E}_i$ generates a submodel which includes $\mathscr{E}_i$ (since the D_α's already constructed form an initial segment of $<_{P,D}$). If $\mathscr{A}$ did not admit a strictly minimal extension, then by Lemma 4 there would be a finitely generated submodel $\mathscr{M}$ of $\mathscr{A}$ which includes $\mathscr{E}_1$ but is not element generated. But $\mathscr{M}$ is the embedded image of one of the D_α-prod (again since they form an initial segment) and so $\mathscr{M}$ must be element generated by the induction hypothesis. This contradiction proves the claim.

By the induction hypothesis, each of the $\mathscr{E}_i$ is element generated and if $i < j$, then $S_{\alpha_i} \subsetneq S_{\alpha_j}$. By Theorem 8, $\mathscr{A}$ admits a strictly minimal extension $\mathscr{B}$ which is single-skied and element generated. Let D_λ be an ultrafilter such that D_λ-prod is isomorphic to $\mathscr{B}$; then D_λ is a P point and is a s.m.u.b. for $\{D_\alpha\colon \alpha < \lambda\}$. Finally, if $\alpha < \lambda$, it is easy to check that $S_{\alpha+1} \subseteq S_\lambda$ (since the map q_λ factors through $q_{\alpha+1}$) and so $S_\alpha \subsetneq S_\lambda$, showing that D_λ satisfies (4).

This completes the inductive construction of $\{D_\alpha\colon \alpha < \aleph_1\}$, and with it, the proof of the theorem.

COROLLARY 10 (CH). *There exist initial segments of the RK ordering, restricted to (isomorphism classes of) P points, of order type* $\aleph_1$.

PROOF. This is simply Theorem 9 with D an RK-minimal ultrafilter. The existence of such D follows from CH (see **[Pu]**, for example); ultrafilters which are minimal in RK are also called *selective* or *Ramsey*, and they are all P points.

COROLLARY 11 (CH). *There exist P points with countably many constellations; in fact, for any countable ordinal* α, *there exist P points E such that the initial segment of RK determined by E has order type* α.

PROOF. Immediate by the previous corollary.

COROLLARY 12 (CH). *For any P point D, there exist initial segments T of* $<_{P,D}$ *such that T is a tree with* $\aleph_1$ *levels, each node has* $2^{\aleph_1}$ *immediate successors, and each countable increasing sequence in T has a unique upper bound in T.*

PROOF. The proof is the same as the proof of Theorem 9, except at successor stages, use Theorem 7 to generate $2^{\aleph_1}$ immediate successors.

6. Two questions. We conclude with two natural questions.

1. Given CH, is there an RK-increasing ω-sequence of P points which does not admit a s.m.u.b. which is a P point? In other words, do we really need element generated models to prove the theorems here?

2. Can our result extend beyond $\aleph_1$? That is, do there exist (CH) initial segments of $<_{P,D}$ of order type $\aleph_1 + 1$ or perhaps $\aleph_2$? In general, the successor cardinal of the continuum would be the best possible.

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